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PROBLEM OF OPTIMAL CONTROL BY THE NATURAL FREQUENCY OF OSCILLATIONS OF AN ORTHOTROPIC SHELL OF REVOLUTION AND ITS FINITE-DIMENSIONAL APPROXIMATION*

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The questions of optimization in problems of oscillations in orthotropic shells of revolution of variable thickness are studied for the case when the thickness and radius of curvature of the shell generatrix are used as the controls. Restrictions are imposed on the principal oscillation eigenfrequency, thickness, internal volume and other parameters. It is shown that a solution of the problem exists and, that the problem can be approximated by a sequence of the finite-dimensional problems. Certain questions of the optimal control in the problem concerning the oscillations of plates of variable thickness with the thickness serving as the control, were studied in /1-4/.

1. Basic assumptions.

Let Ω be a rectangular region in $R^2: \Omega = \{(\varphi, z) \mid 0 < \varphi < 2\pi,\ 0 < z < L\};\ H_0 = W_{2,0}{}^1(\Omega) \times W_{2,0}{}^1(\Omega) \times W_{2,0}{}^2(\Omega)$ is a direct product of the Sobolev spaces /5/ of functions 2π-periodic in φ, $H_0 = \{\omega = (u, v, w) \mid u, v \in W_{2,0}{}^1(\Omega),\ w \in W_{2,0}{}^2(\Omega)\}$ ($W_{2,0}{}^l(\Omega)$ a subspace of the space $W_2{}^l(\Omega)$ with the norm $W_2{}^l(\Omega)$). We denote by H the closure on the norm

$$\|\omega\|_H{}^2 = \|u\|^2_{W_2{}^1(\Omega)} + \|v\|^2_{W_2{}^1(\Omega)} + \|w\|^2_{W_2{}^2(\Omega)} \tag{1.1}$$

of a set of functions $\omega \in H_0$ periodic in φ, infinitely differentiable in the strip $0 < z < L$, $-\infty < \varphi < \infty$ and satisfying the boundary conditions of the problem in question /6/. We introduce the set

$$U = \{t = (h, r) \mid h \in C(\bar\Omega),\ \ r \in C^3[0, L],\ \ e_1 \leqslant h \leqslant e_2,\ e_3 \leqslant r \leqslant e_4\} \tag{1.2}$$

where e_i are positive constants fitted with a topology generated by the product of strong topologies of the spaces $C(\bar\Omega)$ and $C^3[0, L]$. We further define on $H \times H$ the families of bilinear symmetric forms $a_t(\omega', \omega'')$ and $b_t(\omega', \omega'')$, depending on the parameter $t \in U$:

$$a_t(\omega', \omega'') = \int_\Omega \{D_1[E_1\varepsilon_1'\varepsilon_1'' + \nu_2E_1(\varepsilon_1'\varepsilon_2'' + \varepsilon_1''\varepsilon_2') + E_2\varepsilon_2'\varepsilon_2'' + (1 - \nu_1\nu_2)G\varepsilon_3'\varepsilon_3''] + D_2[E_1\varepsilon_4'\varepsilon_4'' + \nu_2E_1(\varepsilon_4'\varepsilon''_5 + \varepsilon_4''\varepsilon_5') + E_2\varepsilon_5'\varepsilon_5'' + 4(1 - \nu_1\nu_2)G\varepsilon_6'\varepsilon_6'']\}A_1{}^2r\,d\Omega \tag{1.3}$$

$$b_t(\omega', \omega'') = \int_\Omega \rho h(u'u'' + v'v'' + w'w'')A_1{}^2r\,d\Omega; \quad \rho = \text{const} > 0$$

$$\varepsilon_1(\omega, r) = \frac{1}{A_1{}^2}\frac{\partial u}{\partial z} - \frac{w}{R_1}, \quad \varepsilon_2(\omega, r) = \frac{1}{r}\frac{\partial v}{\partial\varphi} + \frac{r'}{A_1{}^2r}u - \frac{w}{R_2} \tag{1.4}$$

$$\varepsilon_3(\omega, r) = \frac{1}{A_1{}^2}\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial u}{\partial\varphi} - \frac{r'}{A_1{}^2r}v,$$

$$\varepsilon_4(\omega, r) = -\frac{1}{A_1{}^2}\frac{\partial}{\partial z}\left(\frac{1}{A_1{}^2}\frac{\partial w}{\partial z} + \frac{u}{R_1}\right)$$

$$\varepsilon_5(\omega, r) = -\frac{1}{r}\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\frac{\partial w}{\partial\varphi} + \frac{v}{R_2}\right) - \frac{r'}{A_1{}^2r}\left(\frac{1}{A_1{}^2}\frac{\partial w}{\partial z} + \frac{u}{R_1}\right)$$

$$\varepsilon_6(\omega, r) = -\frac{1}{A_1{}^2r}\left(\frac{\partial^2 w}{\partial\varphi\,\partial z} - \frac{r'}{r}\frac{\partial w}{\partial z}\right) - \frac{1}{R_1r}\frac{\partial u}{\partial\varphi} - \frac{1}{R_2A_1{}^2}\left(\frac{\partial v}{\partial z} - \frac{r'}{r}v\right)$$

$$r' = \frac{dr}{dz}, \quad A_1{}^2 = (1 + r'^2)^{1/2}, \quad R_1 = \left(\frac{d^2r}{dz^2}\right)^{-1}A_1{}^6, \quad R_2 = -rA_1{}^2, \quad D_1(h) = \frac{h}{1 - \nu_1\nu_2}, \quad D_2(h) = \frac{h^3}{12(1 - \nu_1\nu_2)}$$

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Here ε_i', ε_i'' are the shell of revolution deformation components /7/ generated by the displacements of the middle surface $\omega', \omega'' \in H$ and depending on the radius $r(z) \in C^3[0, L]$ of the generatrix, the coefficients D_1 and D_2 depend on the shell thickness $h(z,\varphi) \in C(\bar{\Omega})$, E_i, ν_i, G are the moduli of elasticity, Poisson's ratios and the shear modulus respectively. In addition $E_1\nu_2 = E_2\nu_1$, L is the length of the shell and ρ in the material density. We introduce the following assumptions:

1) E_i, ν_i, G are positive constants while $|\nu_i| < 1$, $i = 1, 2$;

2) the conditions $\varepsilon_i(\omega, r) = 0$ $(i = 1, 2, \ldots, 6)$ imply at any $t \in U, \omega \in H$, that $\omega = 0$.

The assumption 1) holds for orthotropic materials, and 2) holds for the case when the shell shows no rigid displacements, i.e. it is clamped so that zero deformations imply zero displacements (see /6/ for more detail).

As in /6/, we can show that when the assumptions 1 and 2 hold, the form $a_t(\omega', \omega'')$ generates in H a scalar product and a norm equivalent to the norm (1.1), i.e. the following inequalities hold:

$$m_{1t}\|\omega\|_H^2 \leqslant a_t(\omega,\omega) \leqslant M_{1t}\|\omega\|_H^2, \quad \forall\omega \in H. \quad \forall t \in U \tag{1.5}$$

where m_{1t} and M_{1t} are positive constants depending on t. It is also clear that the form $b_t(\omega', \omega'')$ generates in H a scalar product and a norm equivalent to the norm of the space $H_b = (L_2(\Omega))^3$:

$$m_2\|\omega\|_{H_b}^2 \leqslant b_t(\omega,\omega) \leqslant M_2\|\omega\|_{H_b}^2, \quad \forall\omega \in H, \quad \forall t \in U; \tag{1.6}$$
$$m_2, M_2 = \text{const} > 0$$

2. Problem of the eigenfrequencies and the forms of shell oscillations. We shall consider the following eigenvalue problem:

$$a_t(\omega, \omega') = \lambda b_t(\omega, \omega'), \quad \forall\omega' \in H \tag{2.1}$$

Taking into account the relations (1.5) and the compactness of the inclusion

$$W_2^1(\Omega) \times W_2^1(\Omega) \times W_2^2(\Omega) \to (L_2(\Omega))^3$$

we obtain, from the known results /8/, the following theorem.

Theorem 1. Let the assumptions 1 and 2 hold. Then for any $t \in U$ the spectral problem (2.1) has a sequence of nonzero solutions $\omega_k \in H$ corresponding to a sequence of eigenvalues λ_k such that $a_t(\omega_k, \omega) = \lambda_k b_t(\omega_k, \omega), \forall\omega \in H, 0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$. Moreover,

$$\lambda_k = \inf\left\{\frac{a_t(\omega,\omega)}{b_t(\omega,\omega)}\,\middle|\, \omega \in H,\ \omega \neq 0,\ b_t(\omega,\omega_i) = 0,\ 1 \leqslant i \leqslant k-1\right\} \tag{2.2}$$

The problem (2.1) is connected with the determination of the eigenfrequencies and types of oscillation of the variable thickness, orthotropic shells of revolution, satisfying the certain clamping conditions which ensure that assumption 2 holds /6/.

3. Infinite-dimensional problem of optimal control. It is clear that the fundamental eigenfrequency λ_1, the corresponding modes of the oscillations ω_1 and the weight of the shell P all depend on the parameter $t = (h, r)$. Denoting these relations by λ_t, ω_t and P_t and taking (2.2) into account, we obtain

$$\lambda_t = \frac{a_t(\omega_t,\omega_t)}{b_t(\omega_t,\omega_t)} = \inf_{\substack{\omega\in H\\ \omega\neq 0}} \frac{a_t(\omega,\omega)}{b_t(\omega,\omega)}; \quad P_t = \int_\Omega \rho A_1^2 rh\, d\Omega \tag{3.1}$$

We shall use t as a control parameter to obtain the minimum weight of the shell P_t, so that the fundamental eigenfrequency λ_t does not fall below a given frequency $\lambda\_$ when the shell thickness h and radius r of the generatrix are bounded from above and below. In this connection, we introduce here the space

$$V = C(\bar{\Omega}) \times C^3[0, L] = \{t = (h, r) \mid h \in C(\bar{\Omega}),\ r \in C^3[0, L]\}$$

Let E be a reflexive Banach space such that $E \subset V$ and the inclusion of E in V is compact. In particular, we can choose E in the form

$$E = W_{p_1}^1(\Omega) \times W_{p_2}^4(0, L) \quad (p_1 > 2,\ p_2 > 1)$$

We define the admissible set of controls by the expression

$$U_\partial = \{t = (h, r) \mid t \in E, \|t\|_E \leqslant C, \ h_- \leqslant h \leqslant h_+, \ r_- \leqslant r \leqslant r_+, \tag{3.2}$$
$$\lambda_- \leqslant \lambda_t, \ \psi_j(t, \omega_t) \leqslant 0, \ j = 1, 2, \ldots, l\}; \quad e_1 < h_- < h_+ < e_2, \ e_3 < r_- < r_+ < e_4$$

with $(t, \omega) \to \psi_j(t, \omega)$ denoting the continuous mapping of $U \times H$ into R (with the topology generated by the product of strong topologies of the spaces $C(\bar{\Omega})$, $C^3[0, L]$ and H). Here C, h_-, h_+, r_-, r_+ are positive constants and e_i are the constants given by (1.2).

The problem of optimal control is to find a function $t_0 = (h_0, r_0)$ such, that

$$t_0 \in U_\partial, \quad P_{t_0} = \inf_{t \in U_\partial} P_t = \inf_{t \in U_\partial} \int_\Omega \rho A_r^2 r h \, d\Omega \tag{3.3}$$

We note that the inequalities $\psi_j(t, \omega_t) \leqslant 0$ restrict other parameters of the orthotropic shell of revolution, depending on the problem in question. For example, in the case when the minimum internal volume V_∂ of the shell of revolution is restricted, we have

$$\psi_1 = V_\partial - \frac{1}{2} \int_\Omega \left(r - \frac{h}{2}\right)^2 d\Omega \tag{3.4}$$

Lemma. The function $t \to \lambda_t$ (3.1) represents a continuous mapping from U, defined by (1.2), into R.

Proof. Let $t_0 = (h_0, R_0)$ be any element belonging to the sequence $U, \{t_n\} = \{(h_n, r_n)\}$ of elements such that

$$t_n \in U, \ t_n \to t_0 \text{ in } U \tag{3.5}$$

We introduce the following notation for $n = 0, 1, 2, \ldots$

$$\lambda^{(n)} = \lambda_{t_n}, \quad \omega_n = \omega_{t_n}, \quad a_n(\omega', \omega'') = a_{t_n}(\omega', \omega''), \quad b_n(\omega', \omega'') = b_{t_n}(\omega', \omega'') \tag{3.6}$$

From (1.3), (1.4), (3.5) and (3.6) we have

$$|a_n(\omega, \omega) - a_0(\omega, \omega)| \leqslant c_n' \|\omega\|_H^2; \quad |b_n(\omega, \omega) - b_0(\omega, \omega)| \leqslant c_n'' \|\omega\|_{H_b}^2 \tag{3.7}$$
$$\forall \omega \in H, \ c_n' \to 0 \text{ and } c_n'' \to 0 \text{ as } n \to \infty$$

and the following inequalities follow from (1.5), (1.6) and (3.7):

$$m_1 \|\omega\|_H^2 \leqslant a_n(\omega, \omega) \leqslant M_1 \|\omega\|_H^2, \quad \forall \omega \in H, \quad n = 0, 1, 2, \ldots, m_1, \quad M_1 = \text{const} > 0 \tag{3.8}$$

Let ω' be any element of H. Then, taking (1.6) and (3.8) into account, we have

$$\frac{m_1 \|\omega'\|_H^2}{M_2 \|\omega'\|_{H_b}^2} \leqslant \frac{a_n(\omega', \omega')}{b_n(\omega', \omega')} \leqslant \frac{M_1 \|\omega'\|_H^2}{m_2 \|\omega'\|_{H_b}^2} = q; \quad n = 0, 1, 2, \ldots \tag{3.9}$$

Taking into account the relations (3.1), (3.6) and (3.9), we obtain

$$\lambda^{(n)} = \inf_{\omega \in Q} \frac{a_n(\omega, \omega)}{b_n(\omega, \omega)}; \quad Q = \left\{\omega \mid \omega \in H, \ \omega \neq 0, \ \frac{\|\omega\|_H^2}{\|\omega\|_{H_b}^2} \leqslant \frac{M_2}{m_1} q\right\}; \qquad n = 0, 1, 2, \ldots \tag{3.10}$$

and from the inequalities (1.6), (3.7) and (3.8) follows

$$\left| \frac{a_n(\omega, \omega)}{b_n(\omega, \omega)} - \frac{a_0(\omega, \omega)}{b_0(\omega, \omega)} \right| \leqslant \varepsilon_n, \quad \forall \omega \in Q, \quad \varepsilon_n \to 0 \text{ as } n \to \infty \tag{3.11}$$

Now, from (3.10) and (3.11) it follows that $\lambda^{(n)} \to \lambda^{(0)}$ as $n \to \infty$, and this completes the proof of the lemma.

Theorem 2. Let the assumptions 1 and 2 hold, and a non-empty set U_∂ be defined by the relation (3.2). Then a solution of the problem (3.3) exists.

Proof. Let the sequence $\{t_n\}_{n=1}^\infty = \{(h_n, r_n)\}_{n=1}^\infty$ be such that

$$t_n \in U_\partial, \lim_{n \to \infty} P_{t_n} = \inf_{t \in U_\partial} P_t \tag{3.12}$$

By virtue of (3.2) we can eliminate, from the sequence $\{t_n\}_{n=1}^\infty$, a subsequence $\{t_m\}_{m=1}^\infty$ such that

$$t_m \in U_\partial, \quad t_m \to t^* \text{ weakly in } E \tag{3.13}$$

Repeating the arguments similar to those used to prove the lemma, we can show that $\lambda^{(m)} \leqslant c_1$, $\| \omega_m \|_H^2 \leqslant c_2$ $(c_1, c_2 = \text{const} > 0)$, and this implies, with the compactness of the inclusion of E into V and the lemma both taken into account, that a subsequence $\{\lambda^{(k)}, \omega_k, h_k, r_k\}_{k=1}^{\infty}$ exists such that

$$\lambda^{(k)} \to \lambda_{h^*} \text{ in } R \tag{3.14}$$

$$\omega_k \to \omega^* \text{ strongly in } H_b \tag{3.15}$$

$$t_k = (h_k, r_k) \to (h^*, r^*) = t^* \text{ weakly in } E \tag{3.16}$$

$$h_k \to h^* \text{ strongly in } C(\bar{\Omega});\ r_k \to r^* \text{ strongly in } C^3[0, L] \tag{3.17}$$

Remembering that $t \to P(t)$ is a continuous mapping of U into R we obtain, from (3.13) and (3.17),

$$P_{t^*} = \lim_{k\to\infty} P_{t_k} = \inf_{t\in U_\partial} P_t;\quad h_- \leqslant h^* \leqslant h_+;\quad r_- \leqslant r^* \leqslant r_+ \tag{3.18}$$

From the relations (3.13) and (3.16) we obtain

$$C \geqslant \underline{\lim}_{k\to\infty} \| t_k \|_E \geqslant \| t^* \|_E \tag{3.19}$$

where C is a constant given by (3.2), and from (3.2), (3.13), (3.15), (3.16) we have

$$0 \geqslant \lim_{k\to\infty} \psi_j(t_k, \omega_k) = \psi_j(t^*, \omega^*);\quad j = 1, \ldots, l \tag{3.20}$$

Taking into account (3.12) and (3.14) we find that $\lim \lambda^{(k)} = \lambda_{h^*} \geqslant \lambda_-$ as $k \to \infty$ and this, together with (3.18)—(3.20), implies that the function $t_0 = t^* = (h^*, r^*)$ is a solution of the problem (3.3).

4. Approximate solution of the problem (3.3). Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of finite-dimensional subspaces in E. The finite-dimensional optimal control problem is to find the function $t_n = (h_n, r_n)$ such, that

$$t_n \in E_n \cap U_\partial;\quad P_{t_n} = \inf_{t\in E_n \cap U_\partial} P_t \tag{4.1}$$

Using the lemma, we can prove the statement (*).

Theorem 3. Let the conditions of Theorem 2 hold, $\{E_n\}$ be a sequence of finite-dimensional subspaces in E satisfying the condition of limiting density

$$\lim_{n\to\infty} \inf_{t\in E_n} \| t - y \|_E = 0,\quad \forall y \in E \tag{4.2}$$

and let a sequence $\{q_n\}_{n=1}^{\infty}$, exist such that $q_n \in U_\partial^0$, $q_n \to t_0$ in E where U_∂^0 denotes the inside of U_∂ and t_0 is the solution of the problem (3.3). Then n_0 exists such that when $\forall n \geqslant n_0$, then the set $E_n \cap U_\partial$ is non-empty, the problem (4.1) has a solution $t_n = (h_n, r_n)$ and

$$\lim_{n\to\infty} P_{t_n} = P_{t_0} = \inf_{t\in U_\partial} P_t$$

We can separate from the sequence $\{t_n\}_{n=n_0}^{\infty}$ a subsequence $\{t_m\}_{m=1}^{\infty}$, such, that $t_m \to t_0$ strongly in V.

A tensor product of the spline spaces /9/ can be used as an example of the finite-dimensional subspaces E_n satisfying the condition (4.2). We note that another approach which does not require that a sequence $\{q_n\}_{n=1}^{\infty}$ exists is available for constructing approximate solutions of the problem (3.3). As in /4/, we can also consider a dual optimal control problem, i.e., the problem of maximizing the fundamental oscillation eigenfrequency of a shell of revolution, with constraints imposed on its weight and other parameters.

REFERENCES

1. ARMAN Zh.-L.P., LUR'E K.A. and CHERKAEV A.V., On solving the eigenvalue optimization problems arising in designing elastic structures. Izv. Akad. Nauk SSSR, MTT, No.5. 1978.
2. BANICHUK N.V., Optimization of the Form of Elastic Bodies. Moscow, NAUKA, 1980.
3. GURA N.M. and SEIRANIAN A.P., Optimal circular plate with constraints imposed on its rigidity and oscillation eigenfrequency. Izv. Akad. Nauk SSSR, MTT, No.1, 1977.
4. LITVINOV V.G., Problem of optimal control of the eigenfrequency in a plate of variable thickness. J. USSR. Comp. Math. math. Phys., Pergamon Press, Vol.19,No.4, 1979.

*) Litvinov V.G. Optimal control of the coefficients in elliptical systems. Preprint In-ta matem. Akad. Nauk USSR, Kiev, No.794, 1979.

5. BESOV O.V., IL'IN V.P. and NIKOL'SKII S.M., Integral Representations of Functions and the Imbedding Theorems. Moscow, NAUKA, 1975.
6. LITVINOV V.G. and MEDVEDEV N.G., Certain problems of stability of the shells of revolution. In book: Matematicheskaia fizika, Ed.26, Kiev, NAUKOVA DUMKA, 1979.
7. GRIGORENKO Ia.M., Isotropic and Anisotropic Layered Shells of Revolution of variable Ridigity. Kiev, NAUKOVA DUMKA, 1973.
8. MIKHLIN S.G., Variational Methods in Mathematical Physics. English translation, Pergamon Press, Book No. 10146, 1964.
9. VARGA R., Functional Analysis and Approximation Theory in Numerical Analysis. Moscow, MIR, 1974.

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